The SOR method

Example

Consider a linear system $Ax = b$, where

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \quad , \quad b = \begin{bmatrix} -1 \\ 7 \\ -7 \end{bmatrix}$$

a) Check, that the SOR method with value $\omega = 1.25$ of the relaxation parameter can be used to solve this system.
b) Compute the first iteration by the SOR method starting at the point $x^{(0)} = (0, 0, 0)^T$.

Solution:

a) Let us verify the sufficient condition for using the SOR method. We have to check, if matrix A is symmetric, positive definite (*spd*): A is symmetric, so let us check positive definitness:

$$\det(3) = 3 > 0 \quad , \quad \det \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} = 8 > 0 \quad , \quad \det \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} = 20 > 0$$

All leading principal minors are positive and so the matrix A is positive definite.

We know, that for *spd* matrices the SOR method converges for values of the relaxation parameter ω from the interval $0 < \omega < 2$.

Conclusion: the SOR method with value $\omega = 1.25$ can be used to solve this system.

b) The iterations of the SOR method are easier to compute by elements than in the vector form:

1. Write the system as equations:

$$\begin{aligned} 3x_1 - x_2 + x_3 &= -1 \\ -x_1 + 3x_2 - x_3 &= 7 \\ x_1 - x_2 + 3x_3 &= -7 \end{aligned}$$

2. First, write down the equations for the *GS* iterations:

$$\begin{aligned} x_1^{(k+1)} &= (-1 + x_2^{(k)} - x_3^{(k)})/3 \\ x_2^{(k+1)} &= (7 + x_1^{(k+1)} + x_3^{(k)})/3 \\ x_3^{(k+1)} &= (-7 - x_1^{(k+1)} + x_2^{(k+1)})/3 \end{aligned}$$

3. Now multiply the right hand side by the parameter ω and add to it the vector $x^{(k)}$ from the previous iteration multiplied by the factor of $(1-\omega)$:

$$\begin{aligned}
x_1^{(k+1)} &= (1-\omega)x_1^{(k)} + \omega(-1 + x_2^{(k)} - x_3^{(k)})/3 \\
x_2^{(k+1)} &= (1-\omega)x_2^{(k)} + \omega(7 + x_1^{(k+1)} + x_3^{(k)})/3 \\
x_3^{(k+1)} &= (1-\omega)x_3^{(k)} + \omega(-7 - x_1^{(k+1)} + x_2^{(k+1)})/3
\end{aligned}$$

4. For $k = 0, 1, 2, \ldots$ compute $x^{(k+1)}$ from these equations, starting by the first one.

Computation for $k = 0$:

$$x_1^{(1)} = (1-\omega)x_1^{(0)} + \omega(-1+x_2^{(0)}-x_3^{(0)})/3 = (1-1.25)\cdot 0 + 1.25\cdot(-1+0-0)/3 = -0.41667$$

$$x_2^{(1)} = (1-\omega)x_2^{(0)} + \omega(7+x_1^{(1)}+x_3^{(0)})/3 = -0.25\cdot 0 + 1.25\cdot(7-0.41667+0)/3 = 2.7431$$

$$x_3^{(1)} = (1-\omega)x_3^{(0)} + \omega(-7-x_1^{(1)}+x_2^{(1)})/3 = -0.25\cdot 0 + 1.25\cdot(-7+0.41667+2.7431)/3 = -1.6001$$

The next three iterations are

$x^{(2)} = (1.4972, 2.1880, -2.2288)^T$,
$x^{(3)} = (1.0494, 1.8782, -2.0141)^T$,
$x^{(4)} = (0.9428, 2.0007, -1.9723)^T$,

the exact solution is equal to $x = (1, 2, -2)^T$.